Exploring a Condition on Multigraphs

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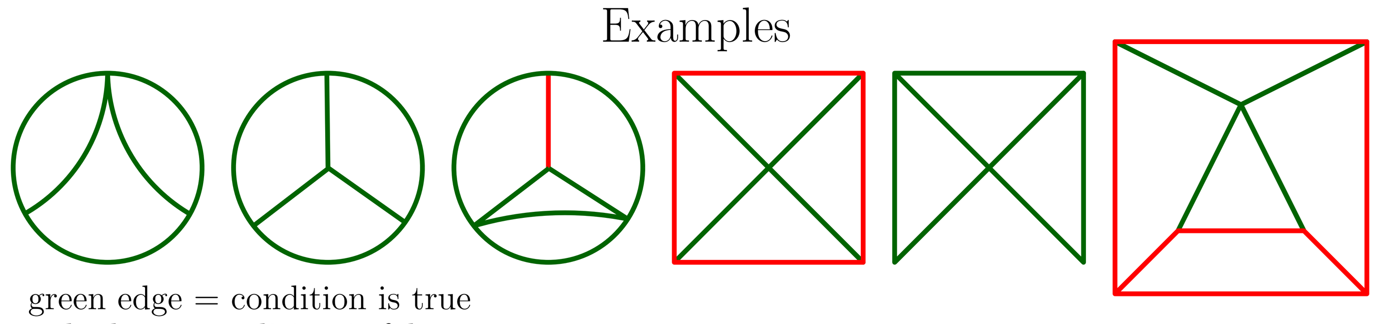
What does this condition look like and why am I studying it?

I am exploring a condition on the edges of multigraphs. The exact statement will be given after I introduce the math, but here is what it looks like on a few graphs:

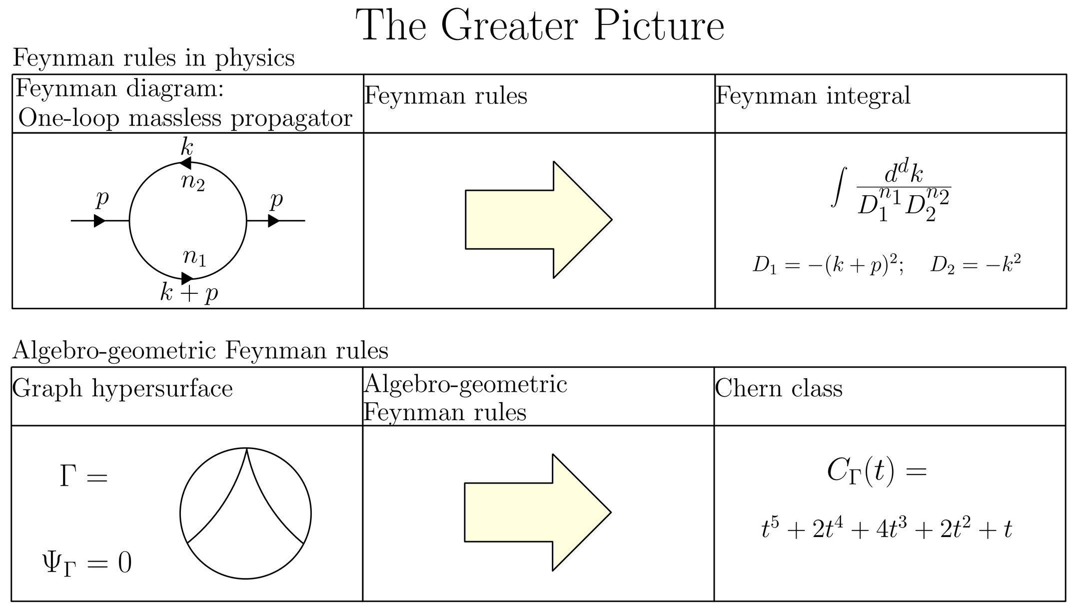
Examples



green edge = condition is true
red edge = condition is false

The condition has ties to research into inventing algebro-geometric Feynman rules (it is part of a criterion guaranteeing a double edge formula for the Chern class of a graph hypersurface.)[1][2] It is also intrinsically interesting.

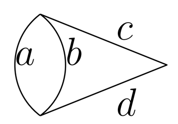
The Greater Picture



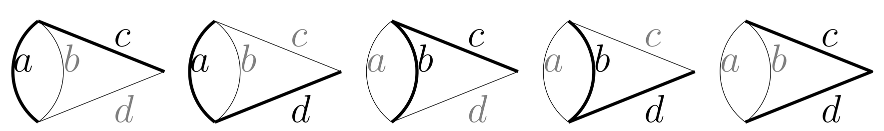
What math is used to express the condition?

The dual-Kirchoff Polynomial

The running example:

$G \;=$ 

The spanning trees of G:



DEFINITION: The dual-Kirchoff polynomial Ψ_G of a connected multigraph G is the sum over the spanning trees T_i of the products of edges a_j not in T_i.[3][4]

$$\Psi_G = \sum_{T_i} \prod_{a_j \notin T_i} a_j$$

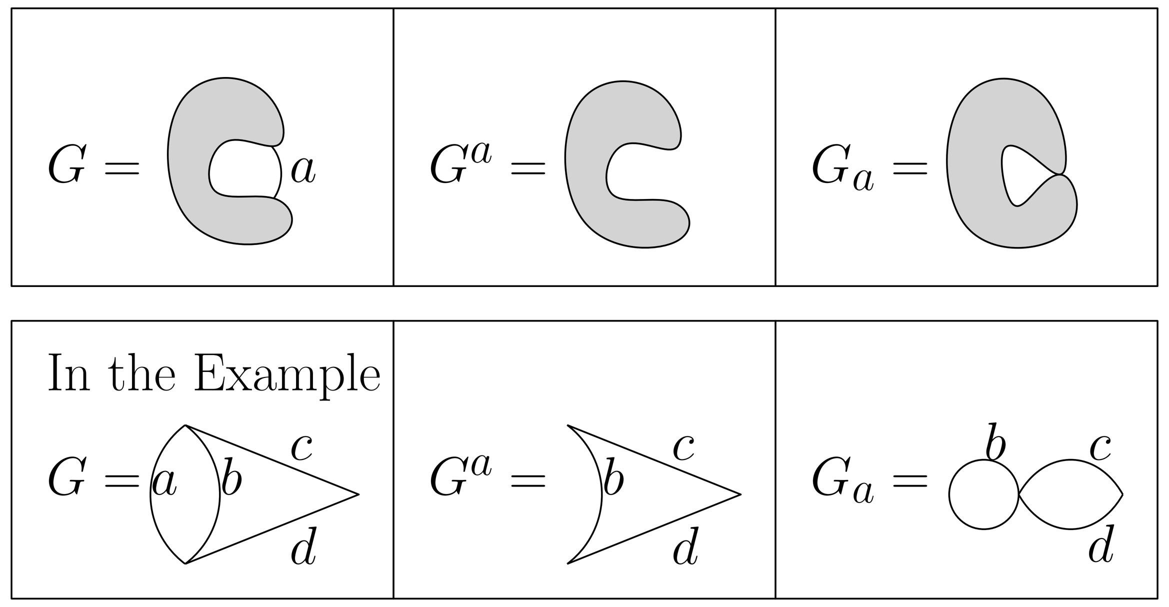
Each spanning tree gives a monomial of Ψ_G. In the example

$$\Psi_G = bd + bc + ad + ac + ab.$$

Edge Deletion and Contraction

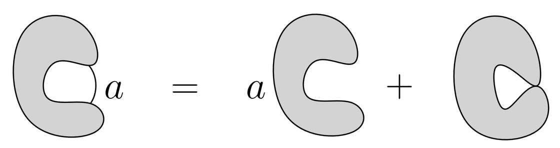
NOTATION:

$$G^a \equiv G \text{ delete } a$$
$$G_a \equiv G \text{ contract } a$$



The dual-Kirchoff polynomial satisfies a deletion-contraction relation:

$$\Psi_G = a\Psi_{G^a} + \Psi_{G_a}$$



Deleting an edge from a graph is the same as differentiating the dual-Kirchoff polynomial; contracting an edge is the same as setting the edge variable to zero.

$$\Psi_{G^a} = \partial_a \Psi_G$$
$$\Psi_{G_a} = (\Psi_G)_{a=0}$$

The Ideal Generated by a Set of Polynomials

DEFINITION: In a polynomial ring R, the ideal $I \subseteq R$ generated by a set of polynomials $S = \{Q_i\}$ is denoted with angled brackets $I \equiv \langle Q_i \rangle$, and

$$\langle Q_i \rangle = \{\Sigma_i P_i Q_i : P_i \in R; Q_i \in S\}.$$

An ideal has closure properties

$$J + K \in I \quad for \quad J, K \in I$$
$$PJ \in I \quad for \quad P \in R,\ J \in I.$$

Statement of the Condition

The condition (G, e) is true when Ψ_G is in the ideal of partial derivatives of Ψ_{G^e}:

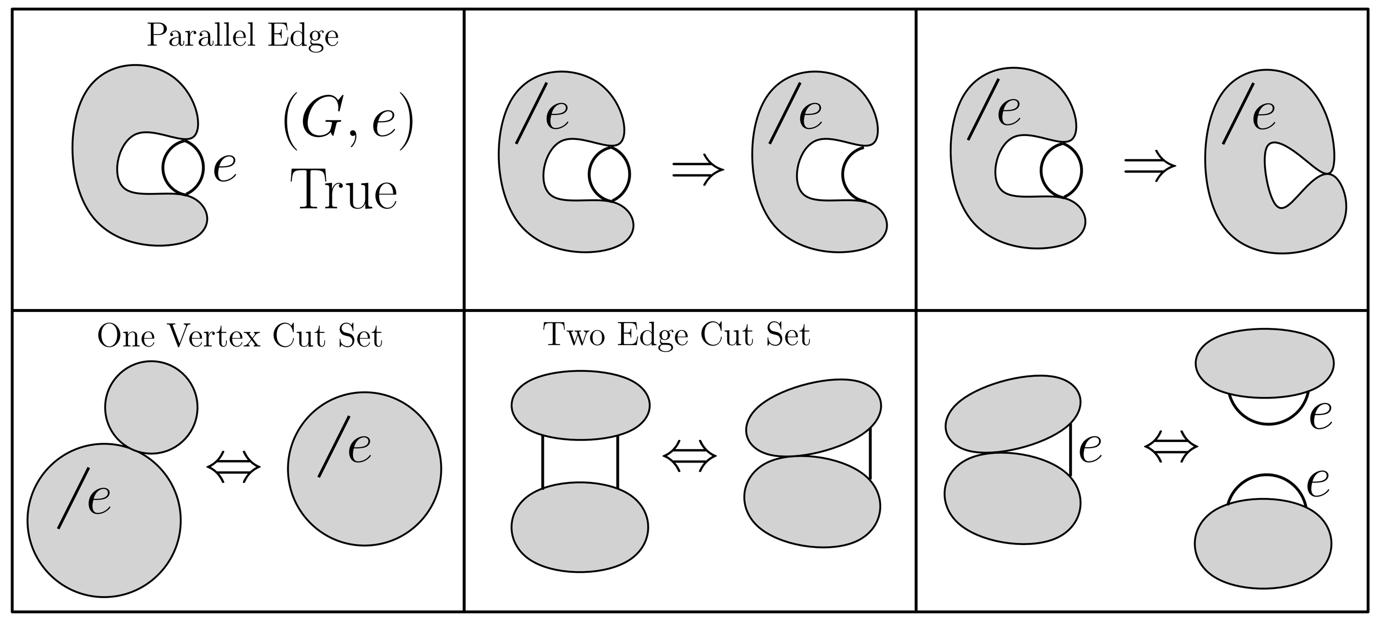
$$\Psi_G \in \langle \partial_{a_i} \Psi_{G^e} \rangle$$

This means there exist polynomials P_i such that

$$\Psi_G = \Sigma_i P_i \partial_{a_i} \Psi_{G^e}$$

Some Properties we Know About

The condition (G, e) is true when e has a parallel edge in G.[1] Parallel edges have interesting properties and play an important role. The condition also has nice invariance properties for graphs with one vertex cut sets and two edge cut sets.



Current Lines of Investigation

Computer Search

I am doing a computer search of small graphs. This is the method used in practice to check the condition:

(1) Calculate the dual-Kirchoff polynomial as

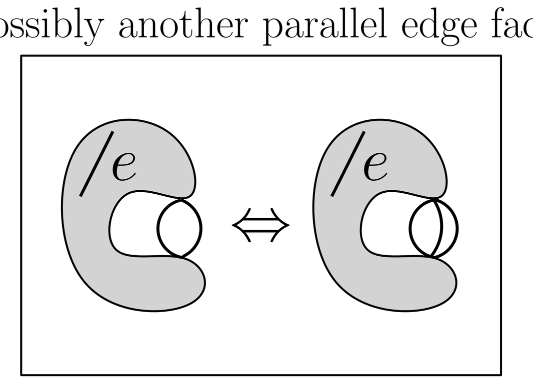
$$\Psi = \det\left(\begin{bmatrix} \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_m \end{matrix} & E^T \\ -E & 0 \end{bmatrix}\right),$$

where E is the oriented incidence matrix with one row removed (this is a consequence of the matrix tree theorem).[3]

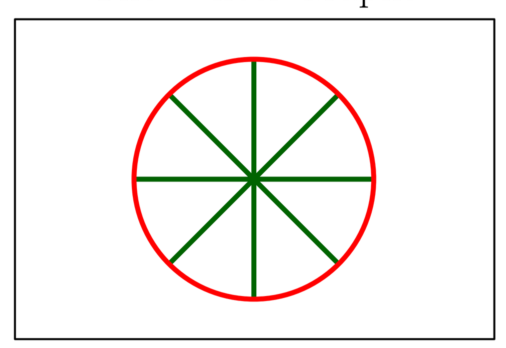
(2) Test for the membership $\Psi_{G_e} \in \langle \partial_{a_i} \Psi_{G^e} \rangle$ by reducing Ψ_{G_e} in the Grobner basis of $\langle \partial_{a_i} \Psi_{G^e} \rangle$ with total degree ordering. The condition is true if Ψ_{G_e} reduces to zero.

Proving More Results

Possibly another parallel edge fact



The Wheel Graphs



REFERENCES

[1] Paolo Aluffi. Chern Classes of Graph Hypersurfaces and Deletion Contraction Relations. *Mosc. Math. J., 12:4 (2012), 671-700. arXiv:1106.1447*
[2] Matilde Marcoli. Feynman Motives. *World Scientific.* ISBN:978-981-4304-48-1, 2006
[3] Karen Yeats. Some Combinatorial Interpretations in Perturbative Quantum Field Theory. arXiv:1302.0080, 2013
[4] Stefan Weinzierl. Feynman Graphs. arXiv:1301.6918